Linear s-r upper bound for sum of medians.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA and Konstantin Knop, St.Petersburg, Russia.

Prove that in any triangle with medians $m_a, m_b, m_c,$ semiperimeter s and inradius r holds inequality

$$m_a + m_b + m_c \leq 2s - 3\left(2\sqrt{3} - 3\right)r\,.$$

Solution.

First we will prove, important by itself, auxiliary inequality

(1) $m_a m_b \leq \dfrac{2c^2 + ab}{4}$

Proof.

Since $m_a^2 = \dfrac{2(b^2+c^2)-a^2}{4}, m_b^2 = \dfrac{2(c^2+a^2)-b^2}{4}$ then $16\left(\left(\dfrac{2c^2+ab}{4}\right)^2 - m_a^2 m_b^2\right) =$
$(2(b^2+c^2)-a^2)(2(c^2+a^2)-b^2) - (2c^2+ab)^2 = 2\left((a^2-b^2)^2 - c^2(a-b)^2\right) =$
$2(a-b)^2(a+b+c)(a+b-c) \geq 0.$

Using inequality (1) we obtain

$(m_a+m_b+m_c)^2 = m_a^2+m_b^2+m_c^2+2(m_a m_b+m_b m_c+m_c m_a) =$

$$\frac{3(a^2+b^2+c^2)}{4} + 2\sum_{cyc} m_a m_b \leq \frac{3(a^2+b^2+c^2)}{4} + 2\sum_{cyc}\frac{2c^2+ab}{4} =$$

$$\frac{7(a^2+b^2+c^2)+2(ab+bc+ca)}{4} = \frac{14(s^2-r(4R+r))+2(s^2+r(4R+r))}{4} =$$

$4s^2 - 3r^2 - 12Rr$ and, therefore, suffice to prove $4s^2 - 3r^2 - 12Rr \leq \left(2s - 3\left(2\sqrt{3}-3\right)r\right)^2.$

Since $s \leq \dfrac{3\sqrt{3}}{2}R$ we have $\left(2s - 3\left(2\sqrt{3}-3\right)r\right)^2 - (4s^2-3r^2-12Rr) =$

$12r\left(R - s\left(2\sqrt{3}-3\right) - r\left(9\sqrt{3}-16\right)\right) \geq 12r\left(R - \dfrac{3\sqrt{3}}{2}R\cdot\left(2\sqrt{3}-3\right) - r\left(9\sqrt{3}-16\right)\right) =$

$6\left(9\sqrt{3}-16\right)r(R-2r) \geq 0.$